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COHOMOLOGY OF GALOIS EXTENSIONS*

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Introduction

Let E be an extension field of a field K and let G be the group of all automorphisms of E over K. Further let K be the fixed field of G. When E is finite separable normal over K, the theorems of Galois cohomology are classical. When E is infinite separable normal over K, they have been extended by considering G with the Krull topology (it is compact), G^n with the product topology and the n-cochains to be continuous maps from G^n into E (or E^*) with discrete topology. When E is transcendental over K, the Krull topology on G is no longer compact most often not compatible with Galois theory and is discrete whenever E is finitely generated over K. Hence consideration of Galois cohomology with Krull topology becomes very difficult in this case.

However it has been shown in [4] that there is a topology J on G which is compatible with Galois theory, for which translations and inverse are homeomorphisms, which in case E is algebraic over K coincides with the Krull topology and which for a large class of extensions is compact.

Hence, in this paper, we consider G with this topology J, n-cochains to be continuous maps from G^n with the product topology into E (or E^*) with the discrete topology and the corresponding cohomology groups $H^n(G,J,E)$ (or $H^n(G,J,E^*)$).

We prove the following:

Theorem. *If E,K,G is such that $\bar{K}$ (the algebraic closure of K in E) is Galois closed, then*

$$H^n(G,J,E) \simeq H^n(S,\tau_0,\bar{K}) \text{ and}$$

$$H^n(G,J,E^*) \simeq H^n(S,\tau_0,\bar{K}^*) \text{ for all } n \geq 0,$$

where $S = \{\sigma \in G(\bar{K}/K) : \sigma$ is extendable to an automorphism of E over $K\}$ and τ_0 is the Krull topology on S.

When $S = G(\bar{K}/K)$, the cohomology of E,K,G with respect to J reduces to the

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classical cohomology of $\bar{K}$ over K with Krull topology for $G(\bar{K}/K)$. We show that this happens precisely when $(G/G_0, \bar{J})$ is a compact space.

The condition that $(G/G_0, \bar{J})$ is compact is satisfied in the following cases:

1) All extensions E such that (G, J) is compact. This class of extensions is completely characterized in Theorem 2.9. This theorem incidentally spotlights algebraically why the Krull topology is compact in the classical case.

2) E is an algebraically closed extension of K and K is of characteristic zero.

3) E is a Dedekind extension of K i.e. for each intermediate field F, F is the fixed field of some subgroup of G.

4) E is a finitely generated extension of K.

We also prove that

Theorem. *If $\bar{K}$ is Galois closed then always $H^1(G, J, E) = 0$ and $H^1(G, J, E^*) = 1$.*

Section 1 deals with the preliminaries, notations and definitions. In Section 2, we determine when J is compact and work out a few needed basic results of J. In Section 3, the main results are proved.

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1

Throughout this paper E is an extension field of a field K, G is the group of all automorphisms of E over K. Further we assume that E is a Galois extension of K i.e. that K is the fixed field of G.

Definition 1.1. $\bar{K} = \{x \in E : x$ is algebraic over $K\}$ i.e. $\bar{K}$ is the algebraic closure of K in E. $\bar{K}$ is a subfield of E and an extension of K. $\bar{K}$ is said to be Galois closed if $E/\bar{K}$ is Galois i.e. if it is the fixed field of some subgroup of G. $G(\bar{K}/K)$ is the group of all automorphisms of $\bar{K}$ over K.

Definition 1.2. If $x \in E$, we let $G(x) = \{\sigma \in G \mid \sigma(x) \neq x\}$. If $s \in G$, $x \in E$, $sG(x) = \{s\sigma : \sigma \in G(x)\}$. The set of all $sG(x)$, $s \in G$, $x \in E$, forms a sub-base (of open sets) for a topology on G. This is the topology J on G. A base for J consists of sets of the form $s_1 G(x_1) \cap s_2 G(x_2) \cap \cdots \cap s_n G(x_n)$. A net s_α, $\alpha \in D$ a directed set, converges to an element s in (G, J) if and only if whenever $t \in G$ and $x \in E$ and $s(x) \neq t(x)$ then there is a $d_0 \in D$ such that $s_d(x) \neq t(x)$ for all $d \geqslant d_0$.

It has been shown in [4] that for (G, J) translations and inverse are homeomorphisms and that a subgroup S of G is Galois closed (i.e. S is the group of all automorphisms of E over $I(S)$, where $I(S) = \{x \in E : s(x) = x$ for every $s \in S\})$ if and only if S is a closed set in (G, J).

Definition 1.3. If $x \in E$, we let $G_x = \{\sigma \in G \mid \sigma(x) = x\}$. We note that $G_x =$

$G \setminus G(x)$ and that the set of all sG_x, $s \in G$, $x \in E$ forms a sub base of closed sets for J.

Definition 1.4. Let G be a group, $(A, +)$ an abelian group on which G acts on the left. Let τ be some topology on the set G (not necessarily related to the action of G in any way). We then say (G, τ, A) is a transformation system.

Definition 1.5. Two transformation systems (G, τ, A) and (G', τ', A) are said to be isomorphic if there exists a map $\theta : G \to G'$ such that θ is an isomorphism between the groups and is a homeomorphism between the topologies and for each $s \in G$ and $a \in A$ we have $sa = \theta(s)a$.

Definition 1.6. Let (G, τ, A) be a transformation system. We take the discrete topology on A. For each $n \geq 1$, we take on G^n the product topology obtained from τ. We define $C^0(G, \tau, A) = A$.

If $n \geq 1$ we define $C^n(G, \tau, A) = \{f : f$ is a continuous map from G^n into $A\}$.

If $f, g \in C^n(G, \tau, A)$ we define $f + g$ by $(f+g)$ $(s_1, \ldots, s_n) = f(s_1, \ldots, s_n) + g(s_1, \ldots, s_n)$. Then $f + g \in C^n(G, \tau, A)$ and under this operation $+$, $C^n(G, \tau, A)$ becomes an abelian group. If $a \in C^0(G, \tau, A)$ we define $da : G \to A$ by $da(\sigma) = \sigma a - a$. da may not belong to $C^1(G, \tau, A)$. When $n \geq 1$ and $f \in C^n(G, \tau, A)$ we define $df : G^{n+1} \to A$ by

$$(df)(s_1, \ldots, s_{n+1}) = s_1 f(s_2, \ldots, s_{n+1}) + \sum_{i=1}^{n} (-1)^i f(s_1, \ldots, s_i s_{i+1}, \ldots, s_{n+1})$$
$$+ (-1)^{n+1} f(s_1, \ldots, s_n).$$

We again note that df need not belong to $C^{n+1}(G, \tau, A)$. We say an $f \in C^n(G, \tau, A)$ is an n-cocycle if df takes every element of G^{n+1} to 0. When $n \geq 1$, an element $g \in C^n(G, \tau, A)$ is called a n-coboundary if $g \in dC^{n-1}(G, \tau, A)$.

We let $Z^n(G, \tau, A) = \{f \mid f$ is an n-cocycle$\}$.

If $n \geq 1$, $B^n(G, \tau, A) = \{g \mid g$ is an n-coboundary$\}$.

We define $B^0(G, \tau, A) = 0$.

By classical arguments, we have d^2 is the zero map, $Z^n(G, \tau, A)$ is an abelian group, $B^n(G, \tau, A)$ is a subgroup of $Z^n(G, \tau, A)$. We define $H^n(G, \tau, A) = Z^n(G, \tau, A)/B^n(G, \tau, A)$, and call it the nth cohomology group of (G, τ) with coefficients in A.

Notation 1.7. If $(A, +)$ is taken to be $(E, +)$ we write $H^n(G, \tau, E)$ for $H^n(G, \tau, A)$. When $(A, +)$ is taken to be $(E^*, \cdot)$ where $E^* = E \setminus 0$ and $\cdot$ is the multiplication in E, we write $H^n(G, \tau, E^*)$ for $H^n(G, \tau, A)$.

Proposition 1.8. *If the transformation systems (G, τ, A) and (G', τ', A') are isomorphic then for each $n \geq 0$ we have*

$$H^n(G, \tau, A) \simeq H^n(G', \tau', A').$$

Proof. This is trivial and hence omitted.

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In this section we develop some results needed for Section 3.

Theorem 2.1. *For $\bar{K}$ the following hold:*

1) *For each $x \in \bar{K}$, G_x is a closed subgroup of finite index in (G, J).*
2) *$\bar{K}$ is invariant under all automorphisms in G.*
3) *$\bar{K}$ is an algebraic separable normal extension of K.*
4) *Any finite number of elements $x_1, \ldots, x_n$ of $\bar{K}$ is contained in a $K(\alpha)$, $\alpha \in \bar{K}$, such that $K(\alpha)$ is a finite separable normal extension of K.*

Proof. We omit the proof since these are fairly standard.

Theorem 2.2. *Let G_0 be the connected component of the identity in (G, J). Then G_0 is a closed normal subgroup of G. If $\bar{K}$ is Galois closed then G_0 is the group of all automorphisms of E over $\bar{K}$ and $\bar{K}$ is the fixed field of G_0.*

Proof. Since translations and inverse are homeomorphisms for (G, J) we get that G_0 is a closed normal subgroup of G. Let G_1 be the group of all automorphisms of E over $\bar{K}$. For each $x \in \bar{K}$, G_x is a closed subgroup of G of finite index in G and since translations are homeomorphisms it is open in (G, J). Hence $G_0 \subset G_x$ and so $G_0 \subset G_1$. Let us assume now $\bar{K}$ is Galois closed. Then $\bar{K}$ is the fixed field of G_1. We claim G_1 is connected. If not there exist two open sets V_1, V_2, in G such that $G_1 \subset V_1 \cup V_2$, $V_1 \cap G_1 \neq \emptyset$, $V_2 \cap G_1 \neq \emptyset$, and $V_1 \cap V_2 \cap G_1 = \emptyset$. This implies that there exist two basic open sets $s_1 G(x_1) \cap \cdots \cap s_m G(x_m)$, $t_1 G(y_1) \cap \cdots \cap t_n G(y_n)$ each intersecting G_1 and such that

$$G_1 \cap s_1 G(x_1) \cap \cdots \cap s_m G(x_m) \cap t_1 G(y_1) \cap \cdots \cap t_n G(y_n) = \emptyset.$$

Hence $G_1 \subset s_1 G_{x_1} \cup \cdots \cup s_m G_{x_m} \cup t_1 G_{y_1} \cup \cdots \cup t_n G_{y_n}$. Now $G_1 = (s_1 G_{x_1} \cap G_1) \cup \cdots \cup (t_n G_{y_n} \cap G_1)$. If $s_1 G_{x_1} \cap G_1 \neq \emptyset$ then $s_1 G_{x_1} \cap G_1 =$ a coset of $G_{x_1} \cap G_1$ and similarly for other terms. Hence $G_1 =$ a finite union of cosets of some of the subgroups $G_{x_i} \cap G_1$. After successively omitting superfluous terms we can apply Lemma 1.3 [4] or theorems of [2] and obtain that $G_1 =$ a finite union of cosets of a subgroup S where S is an intersection of some of the G_{x_i} and G_1.

Now S is a Galois closed subgroup since G_1 and G_{x_i} are Galois closed. If F is the fixed field of S then F is a finite extension of $\bar{K}$ since S is of finite index in G_1 and $\bar{K}$ is the fixed field of G_1. Hence F is algebraic over $\bar{K}$. But $\bar{K}$ is algebraic over K and hence F is algebraic over K. By the definition of $\bar{K}$ we get $F = \bar{K}$. This implies that $S = G_1$. Hence $G_1 \subset G_{x_i}$ for at least one i. Then $s_i G_{x_i} \cap G_1 \neq \emptyset$ implies $s_i \in G_{x_i}$ and hence $s_i G_{x_i} = G_{x_i}$. So $s_i G(x_i) = G \setminus s_i G_{x_i}$ is disjoint with G_1. This is a contradiction since G_1 intersects each of the basic open sets. Now G_1 is connected, $G_1 \supset G_0$ imply $G_1 = G_0$. Finally since $\bar{K}$ is Galois closed, $\bar{K}$ is the fixed field of G_1 and hence of G_0.

Lemma 2.3. *Let $\bar{K}$ be Galois closed. Let $s_1 G(x_1) \cap \cdots \cap s_n G(x_n)$ be a basic open set disjoint with a coset sG_0, where G_0 is the connected component at the identity of G and such that $(\bigcap_{i \neq j} s_i G(x_i)) \cap sG_0 \neq \emptyset$ for any j. Then $x_i \in \bar{K}$ for each i.*

Proof. We get from the hypothesis that $sG_0 \subset \bigcup_1^n s_i G_{x_i}$ and no term here is redundant. Hence $G_0 \subset \bigcup_1^n (s^{-1}s_i) G_{x_i}$ and hence $G_0 = \bigcup_1^n (s^{-1}s_i G_{x_i} \cap G_0)$. Now $s^{-1}s_i G_{x_i} \cap G_0$ can be written as coset of $G_{x_i} \cap G_0$. Hence G_0 is a finite union of cosets of the groups $G_{x_i} \cap G_0$. By Lemma 1.3 of [4] or [2] we get that G_0 is a finite union of cosets of the group $G_0 \cap G_{x_1} \cap \cdots \cap G_{x_n}$. Now $G_0, G_{x_1}, \ldots, G_{x_n}$ are all closed subgroups under J and hence $G_0 \cap G_{x_i} \cap \cdots \cap G_{x_n}$ is a closed subgroup. Since this of finite index in G_0 and translations are homeomorphisms of for J, we get that $G_0 \cap G_{x_i} \cap \cdots \cap G_{x_n}$ is open in G_0. Since G_0 is connected we get that $G_0 = G_0 \cap G_{x_1} \cdots \cap G_{x_n}$. This implies $G_0 \subset G_{x_i}$ for each i. Since $\bar{K}$ is Galois closed, $\bar{K}$ is the fixed field of G_0 by Theorem 2.2. Hence the fixed field of $G_{x_i} \subset \bar{K}$ and so $x_i \in \bar{K}$ for each i.

Theorem 2.4. *Let $\bar{K}$ be Galois closed. Let V be an open subset of (G, J) containing the identity e of G and such that $G \setminus V = \bigcup_{i \in I} t_i G_0$ for some index set I. Then there is an $\alpha \in \bar{K}$ such that $G_\alpha \subset V$ and $K(\alpha)$ is a finite normal extension of K.*

Proof. Let $s_1 G(x_1) \cap \cdots \cap s_n G(x_n)$ be a basic open set containing the identity e and contained in V such that $\bigcap_{i \neq j} s_i G(x_1) \cap (G \setminus V) \neq \emptyset$ for each j (such a basic open set definitely exists). Then $G \setminus V \subset \bigcup_1^n s_i G_{x_i}$. Then $t_k G_0 \subset \bigcup_1^n s_i G_{x_i}$ for each $k \in I$. By successively omitting the superfluous terms for this $t_k G_0$ we can apply Lemma 2.3 and get that some of the $x_i \in \bar{K}$. By the choice of the basic open set, each $s_i G_{x_i}$ will occur irredudantly for some $t_k G_0$ and hence all the $x_i \in \bar{K}$.

Now G_{x_i} is a open and closed set since $x_i \in \bar{K}$ (by Theorem 2.1 and the fact that translations are homeomorphisms for (G, J)). Since $e \in s_i G(x_i)$ we get that $s_i^{-1} \in G(x_i)$. If $\sigma \in G_{x_i}$ then $\sigma(x_i) = x_i$. Hence $(s_i^{-1}\sigma)(x_i) = s_i^{-1}(x_i) \neq x_i$ and so $s_i^{-1}\sigma \in G(x_i)$ and so $\sigma \in s_i G(x_i)$. Thus $G_{x_i} \subset s_i G(x_i)$. Hence we get that $e \in G_{x_1} \cap \cdots \cap G_{x_n} \subset \bigcap_1^n s_i G(x_i) \subset V$. By Theorem 2.1 there is an $\alpha \in \bar{K}$ such that $K(\alpha)/K$ is finite separable normal and $x_1, \ldots, x_n$ are in $K(\alpha)$. Now $e \in G_\alpha \subset G_{x_1} \cap \cdots \cap G_{x_n} \subset V$.

Definition 2.5. Let S be a group of automorphisms of $\bar{K}$ over K. If $\alpha \in \bar{K}$ let $S_\alpha = \{\sigma \in S : \sigma(\alpha) = \alpha\}$. S_α is a subgroup of finite index in S. If $s \in S$ and $\alpha \in \bar{K}$ let $sS_\alpha = \{s\sigma : \sigma \in S_\alpha\}$. The collection of all sS_α, $s \equiv S$, $\alpha \in \bar{K}$ form a subbase of open sets for a topology τ_0 on S. This topology τ_0 is called the Krull topology on S. τ_0 is translation invariant in S. If S has fixed field K then (S, τ_0) is a Hausdorff topological group and is the subspace of $G(\bar{K}/K)$ with Krull topology.

Theorem 2.6. *Let $\bar{K}$ be Galois closed. Let $\bar{G} = G/G_0$ and $\bar{J}$ be the quotient topology on $\bar{G}$ from (G, J) under the canonical map $\alpha_1 : G \to \bar{G}$. Then $(\bar{G}, \bar{J}, \bar{K})$ is a transformation system isomorphic to the system $(S, \tau_0, \bar{K})$ where $S = \{\sigma \in G(\bar{K}/K) : \sigma$ is extendable to an automorphism of E over $K\}$ and τ_0 is the Krull topology on it. Further $(\bar{G}, \bar{J}, \bar{K}^*)$ is a transformation system isomorphic to the system $(S, \tau_0, \bar{K}^*)$.*

Proof. By Theorem 2.2, G_0 is the group of all automorphisms of E over $\bar{K}$ and $\bar{K}$ is the fixed field of G_0. Also if $x \in \bar{K}$ and $s \in G$ then $s(x) \in \bar{K}$ (by Theorem 2.1). If $sG_0 \in \bar{G}$ and $x \in \bar{K}$ we define $(sG_0)x = s(x)$. If $sG_0 = tG_0$ then $s^{-1}t \in G_0$ and so $s^{-1}t(x) = x$ if $x \in \bar{K}$ and so $s(x) = t(x)$. Hence we easily get that $(\bar{G}, \bar{J}, \bar{K})$ is a transformation system with this definition.

We now define a map $\theta : \bar{G} \to S$ by $\theta(sG_0) = s \mid \bar{K}$. $s \mid \bar{K} \in S$ since $s \in G$. If $sG_0 = tG_0$ then $s^{-1}t \in G_0$ and hence $s^{-1}t$ is identity on $\bar{K}$ and so $s \mid \bar{K} = t \mid \bar{K}$. Hence θ is a well defined map. Moreover θ is injective since if $\theta(sG_0) = \theta(tG_0)$ then $s \mid \bar{K} = t \mid \bar{K}$ and so $s^{-1}t \mid \bar{K} =$ identity and hence $s^{-1}t \in G_0$ (by Theorem 2.2) and thus $sG_0 = tG_0$. If $s \in S$ and σ is an extension of s to an automorphism of E over K then $\theta(\sigma G_0) = \sigma \mid \bar{K} = s$ and hence θ is onto. It is easily seen that θ is a homomorphism and hence θ is an isomorphism. Further for each $x \in \bar{K}$, we have $(sG_0)x = s(x) = (s \mid \bar{K})(x) = \theta(sG_0)x$.

Both for $(\bar{G}, \bar{J})$ and (S, τ_0) translations are homeomorphisms (this is easy to see). Hence to show that θ is continuous it is enough to show that $\theta^{-1}(S_\alpha)$ is open in $(\bar{G}, \bar{J})$ if $\alpha \in \bar{K}$.

Now $\theta^{-1}(S_\alpha)$ is precisely the image of G_α in $\bar{G}$. Since $\alpha \in \bar{K}$, G_α is a closed subgroup containing G_0 of finite index in (G, J) and hence open in (G, J). Hence G_α is the full pre-image of $\theta^{-1}(S_\alpha)$. Since $\bar{J}$ is the quotient topology we have $\theta^{-1}(S_\alpha)$ is open in $\bar{G}$. Hence θ is continuous.

To show that θ is a open map it is enough to show that if V_1 is a open set containing the identity in $(\bar{G}, \bar{J})$ then $\theta(V_1)$ is a neighbourhood of the identity in (S, τ_0).

Let V be the preimage of V_1 in (G, J). Then V is open in (G, J) contains the identity e of G, and $G \setminus V$ is a union of cosets of G_0. Hence by Theorem 2.4, there is an $\alpha \in \bar{K}$ such that $G_\alpha \subset V$. Now $\alpha_1(G_\alpha) \subset V_1$ and $\theta(\alpha_1(G_\alpha)) = S_\alpha$. Thus $\theta(V_1)$ contains S_α and hence is a neighbourhood of the identity in (S, τ_0). Hence θ is a open map. Therefore θ is a homeomorphism.

It follows that the transformation systems $(\bar{G}, \bar{J}, \bar{K})$ and $(S, \tau_0, \bar{K})$ are isomorphic under θ. Since 0 is a fixed point under all the actions here we get that $(\bar{G}, \bar{J}, \bar{K}^*)$ is a transformation system isomorphic to the system $(S, \tau_0, \bar{K}^*)$.

Theorem 2.7. *If $\bar{K}$ is Galois closed then $(\bar{G}, \bar{J})$ is a compact space if and only if every automorphism of $\bar{K}/K$ is extendable to an automorphism of E over K, i.e. $S = G(\bar{K}/K)$.*

Proof. Since each $\sigma \in G$ gives an element $\sigma \mid \bar{K}$ of S, we get that the fixed field of S in $\bar{K}$ is K. Hence S is a dense subgroup of $(G(\bar{K}/K), \tau_0)$. Also (S, τ_0) is the subspace of $(G(\bar{K}/K), \tau_0)$.

If $(\bar{G}, \bar{J})$ is compact, from the proof of Theorem 2.6, since θ is a homeomorphism we get S is compact. Since $(G(\bar{K}/K), \tau_0)$ is Hausdorff S is closed and hence $S = G(\bar{K}/K)$.

Conversely if $S = G(\bar{K}/K)$ then by classical Krull Galois theory $(G(\bar{K}/K), \tau_0)$ is a compact space and again since θ is a honeomorphism, we get $(\bar{G}, \bar{J})$ is a compact space.

This establishes Theorem 2.7.

Theorem 2.8. *If $\bar{K}$ is Galois closed then $(\bar{G}, \bar{J})$ is a locally compact space if and only if $(\bar{G}, \bar{J})$ is a compact space.*

Proof. If $(\bar{G}, \bar{J})$ is compact then surely it is locally compact. Conversely if $(\bar{G}, \bar{J})$ is locally compact then proceeding as in the proof of Theorem 2.7, we get that S is a locally compact dense subgroup of the Hausdorff group $(G(\bar{K}/K), \tau_0)$ and so is open and hence is closed and there by $S = G(\bar{K}/K)$. Then Theorem 2.7 completes the proof.

Theorem 2.9. *For the extension E/K, (G, J) is a compact space if and only if the following condition is satisfied. If F is an intermediate field of E/K and σ is an isomorphism of F/K into E/K such that given any finite number of elements $\alpha_1, \ldots, \alpha_n$ in F there exists an element $s \in G$ such that $\sigma(\alpha_i) = s(\alpha_1)$ for $i = 1, 2, \ldots, n$ then σ can be extended to an automorphism of E over K.*

Proof. Suppose E/K satisfies the above condition. To show that J is compact. By Alexander's theorem it is enough show that any open cover of G by members of the sub-base has a finite subcover. Let $\mathscr{F}$ be a open cover consisting of members from the sub-base.

If $\mathscr{F}$ has two distinct members of the form $s_1 G(x)$ and $s_2 G(x)$ then we claim that $G = s_1 G(x) \cup s_2 G(x)$. For if $s \in G$ and $s \notin s_1 G(x)$ then $s_1^{-1} s \notin G(x)$ and hence $s_1^{-1} s(x) = x$. This implies $s_1(x) = s(x)$. Similarly $s \notin s_2 G(x)$ implies $s(x) = s_2(x)$. Hence if $s \notin s_1 G(x) \cup s_2 G(x)$ then $s_1(x) = s_2(x)$. This implies $s_1 G_x = s_2 G_x$ and hence $s_1 G(x) = s_2 G(x)$. Hence we can suppose that $\mathscr{F} = \{s_x G(x)\}$, $x \in I$, for a suitable index set. Suppose $\mathscr{F}$ does not allow of a finite sub cover. Consider the family of closed sets $\{s_x G_x\} x \in I$. This has the finite intersection property. Hence it is enough to show that $\bigcap_{x \in I} s_x G_x \neq \emptyset$; since this would imply that $\mathscr{F}$ does not cover G.

Let F be the sub field $K(\{x\}_{x\in I})$. If $x_1, x_2, \ldots, x_n$ is any finite sub collection from I, then there exists an $s \in s_{x_1}G_{x_1} \cap \cdots \cap s_{x_n}G_{x_n}$. Hence for any $x_1, \ldots, x_n$ from I there exists an $s \in G$ such that $x_1, \ldots, x_n$ are mapped by s to $s_{x_1}(x_1), \ldots, s_{x_n}(x_n)$ respectively. Hence the correspondence $x \to s_x(x)$ yields an isomorphism of F/K into E/K satisfying the condition mentioned in the theorem. Hence there exists a $s \in G$ extending this isomorphism, i.e. there is a $s \in G$ such that for each $x \in I$, $s(x) = s_x(x)$. This implies $s \in s_x G_x$ for every $x \in I$. This completes the sufficiency part.

Suppose now J is compact. To show that the condition of the theorem is satisfied. Let F be an intermediate field and σ be an isomorphism of F/K into E/K satisfying the condition mentioned in the theorem. To show that σ is extendible to an automorphism of E/K. Let $F = K(\{x\}_{x\in I})$ for a suitable index set. For each $x \in I$ choose an $s_x \in G$ such that $\sigma(x) = s_x(x)$. Now consider the collection of closed set $\{s_x G_x\}_{x\in I}$. The condition in the theorem implies that this collection satisfies the finite intersection property. Hence by the compactness of J, there exists $s \in \bigcap_{x\in I} s_x G_x$. Hence for each $x \in I$, $s(x) = s_x(x) = \sigma(x)$. Hence s extends σ. This completes the proof.

Remark 2.10. a) Let E/K be an algebraic separable normal extension. Let τ_0 be the Krull topology on G. If F is any intermediate field any isomorphism σ of F/K into E/K satisfies the condition mentioned in the theorem. Also τ_0 coincides with J [4]. Hence the compactness of the Krull topology is precisely equivalent to the algebraic fact: any isomorphism $\sigma: F/K \to E/K$ of an intermediate field F is extendible to an automorphism of E/K. This theorem and its proof incidentally give a simpler proof for the verification of the compactness of the Krull topology.

b) When E/K is of finite trancedence degree, Theorem 4.3 in [4] gives a weaker condition for compactness of J. This weaker condition is not sufficient when E/K is of infinite transcendence degree. For if $\mathbb{Q}$ is the rational number field the algebraic closure of $\mathbb{Q}(x_1, x_2, \ldots)$ where x_i are algebraically independent over $\mathbb{Q}$ satisfies the condition but the topology J is not compact in this case.

c) Even if E/K satisfies the condition that any automorphism of an intermediate field F/K can be extended to an automorphism of E/K, J need not be compact. The example in (b) again serves the purpose.

Remark 2.11. It has been proved in [4] that J is compact in the following cases: (1) E is a finitely generated extension on K; (2) E is of finite trancedence degree over K and E is a Dedekind extension of K, in particular when K is of characteristic zero and E an algebraically closed extension of finite transcedence degree. We get also the case when E is any extension of transcedence degree 1 over K and every automorphism of $\bar{K}$ over K can be extended to an automorphism of E over K.

Theorem 2.12. *Let E be a Dedekind extension of K. Then $(\bar{G}, \bar{J})$ is a compact space.*

Proof. If $E = \bar{K}$ there is nothing to prove. Let B be a transcedence base of E over K and let $F = K(B)$. Now let G_1 be the group of all automorphisms of E over F.

Now E is algebraic Dedekind and hence separable normal over F.

If sG_x is a sub-basic closed set in (G, J) and $sG_x \cap G_1 \neq \emptyset$ then $sG_x \cap G_1 = s_1(G_x \cap G_1)$ for any $s_1 \in sG_x \cap G_1$. But $G_x \cap G_1$ is a Galois closed subgroup contained in G_1 and hence a closed subgroup of G_1 under the J topology on G_1 (considering E, F, G_1). On the other hand the J-topology on G_1 is coarser than the subspace topology on G_1 and hence we get that the J-topology on G_1 is equal to the subspace topology on G_1. Since E is algebraic over F by Theorem 2.9 of [4], the J topology on G_1 coincides with the Krull topology on G_1. But under the Krull topology G_1 is compact. Hence G_1 is a compact subset of (G, J). Now if σ is an automorphism of $\bar{K}$ over K, then σ easily extends to an automorphism of $F(\bar{K})$ over F and then since E is algebraic separable normal over F, this extends to an automorphism of E over F. Hence σ extends to an element of G_1. Now using Theorem 2.2, if $sG_0 \in \bar{G}$, and $s_1 \in G_1$ is such that $s_1 \mid \bar{K} = s \mid \bar{K}$ then s_1 maps onto sG_0 in $\bar{G}$. So G_1 maps onto $\bar{G}$. Hence it follows that $(\bar{G}, \bar{J})$ is compact.

Corollary 2.13. *$(\bar{G}, \bar{J})$ is compact in the following cases:*

1) *E is an algebraically closed extension of K and K is of characteristic zero.*
2) *$\bar{K}$ is a finite extension of K.*

Proof. In case (1) E is surely a Dedekind extension of K and hence we can apply Theorem 2.6. In case (2) $\bar{G}$ is a finite space and hence compact.

3

We prove the main results in this section.

Theorem 3.1. *Let $\bar{K}$ be Galois closed. Then for all $n > 0$*

$$H^n(G, J, E) \simeq H^n(\bar{G}, \bar{J}, \bar{K}) \text{ and}$$

$$H^n(G, J, E^*) \simeq H^n(\bar{G}, \bar{J}, \bar{K}^*),$$

where $\bar{G} = G/G_0$, $\bar{J}$ is the quotient topology on $\bar{G}$ from G under the canonical homomorphism $\alpha_1 : G \to \bar{G} = G/G_0$.

Proof. Since translations are homeomorphism for J we get that α_1 is a *continuous open map* from (G, J) onto $(\bar{G}, \bar{J})$. This makes the map (we call it α_n) $(s_1, \ldots, s_n) \to (\alpha_1(s_1), \ldots, \alpha_1(s_n))$ from G^n to $\bar{G}^n$ also a continuous open map when

G^n and $\bar{G}^n$ are given the product topologies. Hence $\bar{G}^n$ with product topology is a quotient of G^n with its product topology (under this map). We will now get a map from $Z^n(G,J,E)$ to $Z^n(\bar{G},\bar{J},\bar{K})$, when $n \geq 1$.

Step (1): Let $f \in C^n(G,J,E)$. f is a continuous map from G^n into E. We define $\bar{f}: \bar{G}^n \to E$ as follows; $\bar{f}(s_1G_0,\ldots,s_nG_0) = f(s_1,\ldots,s_n)$. We note that for G^n also translations are homeomorphisms, G_0^n is a connected subset of G^n, and f is constant on connected subsets since E is discrete.

We get that if $(s_1G_0,\ldots,s_nG_0) = (t_1G_0,\ldots,t_nG_0)$ then $(t_1,\ldots,t_n) \in (s_1,\ldots,s_n)G_0^n$ and hence $f(t_1,\ldots,t_n) = f(s_1,\ldots,s_n)$. This shows that $\bar{f}$ is well defined and we get $f = \bar{f} \circ \alpha_n$. Since $\bar{G}^n$ has quotient topology under α_n and f is continuous, we get $\bar{f}$ is continuous.

Step (2): If $f \in C^n(G,J,E)$ and $df \in C^{n+1}(G,J,E)$ then $s_1f(s_2,\ldots,s_{n+1}) = f(s_2,\ldots,s_{n+1})$ if $s_1 \in G_0$ and $(s_2,\ldots,s_{n+1}) \in G^n$. Hence $f(s_2,\ldots,s_{n+1}) \in \bar{K}$. For

$$df(s_1,\ldots,s_{n+1}) = s_1f(s_2,\ldots,s_n) + \sum_{i=1}^{n} (-1)^i f(s_1,\ldots,s_is_{i+1},\ldots,s_{n+1})$$
$$+ (-1)^{n+1} f(s_1,\ldots,s_n).$$

Since $s_1 \in G_0$, we get that

$$(s_1s_2,\ldots,s_{n+1}) \in G_0^n(s_2,\ldots,s_{n+1})$$
$$(s_1,s_2s_3,\ldots,s_{n+1}) \in G_0^n(1,s_2s_3,\ldots)$$
$$\vdots$$

Hence

$$f(s_1s_2,\ldots,s_{n+1}) = f(s_2,\ldots,s_{n+1})$$
$$f(s_1,s_2s_3,\ldots,s_{n+1}) = f(1,s_2s_3,\ldots,s_{n+1})$$
$$\vdots$$
$$f(s_1,s_2,\ldots,s_n) = f(1,s_2,\ldots,s_n).$$

Hence the above expression becomes

$$df(s_1,\ldots,s_{n+1}) = s_1f(s_2,\ldots,s_{n+1}) - f(s_2,\ldots,s_{n+1}) + df(1,s_2,\ldots,s_{n+1}).$$

But $df \in C^{n+1}(G,J,E)$ implies $df(s_1,\ldots,s_{n+1}) = df(1,s_2,\ldots,s_{n+1})$. This yields $s_1f(s_2,\ldots,s_{n+1}) = f(s_2,\ldots,s_{n+1})$. Hence $f(s_2,\ldots,s_{n+1})$ belongs to the fixed field of G_0. Then Theorem 2.2 completes the proof of step (2).

Step (3): Let now $f \in Z^n(G,J,E)$. Then $f \in C^n(G,J,E)$ and $df = 0$. By step (2) we get $\bar{f}(\bar{G}^n) \subset \bar{K}$. Hence $\bar{f}$ is a continuous map from $\bar{G}^n$ into $\bar{K}$. Also a direct checking from the fact that $df = 0$, yields $d\bar{f} = 0$. Hence $\bar{f} \in Z^n(\bar{G},\bar{J},\bar{K})$.

We define a map $\beta: Z^n(G,J,E) \to Z^n(\bar{G},\bar{J},\bar{K})$ by $\beta(f) = \bar{f}$.

It is easy to see that $\beta(f+g) = \beta(f) + \beta(g)$ and that $f \neq 0$ implies that $\beta(f) \neq 0$. Also if $g \in Z^n(\bar{G},\bar{J},\bar{K})$ we define $g_1: G^n \to E$ by $g_1 = g \circ \alpha_n$. g_1 is continuous since

α_n and g are continuous. Also $dg = 0$ yields easily that $dg_1 = 0$. Furthermore, we verify easily that $\beta(g_1) = g$. Hence β is surjective. It follows that β is an isomorphism.

Step (4): We now show that $\beta(B^n(G,J,E)) = B^n(\bar{G},\bar{J},\bar{K})$.

Case (i): $n = 1$. If $f \in B^1(G,J,E)$, $f = dg$, $g \in E$ and so $f(s) = sg - g$ for each $s \in G$. Now $f(1) = 0$ and hence $f(G_0) = 0$ and so g is fixed by each element of G_0 and hence $g \in \bar{K}$. Therefore $f(s) = (sG_0)g - g$ and so it follows that $\beta(f) = \bar{f} = dg$, $g \in \bar{K}$ and hence $\beta(f) \in B^1(\bar{G},\bar{J},\bar{K})$.

If $h \in B^1(\bar{G},\bar{J},\bar{K})$, then $h = dg$ for a $g \in \bar{K}$. Consider now g as an element of E and let $h_1 = dg$. If $s \in G$, $h_1(s) = sg - g = (sG_0)g - g = (h \circ \alpha_1)(s)$ and hence $h_1 = h \circ \alpha_1$. So $h_1 \in C^1(G,J,E)$ and hence $h_1 \in B^1(G,J,E)$. Now $\beta(h_1)(sG_0) = h_1(s) = sg - g = (sG_0)g - g = h(sG_0)$ and so $\beta(h_1) = h$. Thus $\beta(B^1(G,J,E)) = B^1(\bar{G},\bar{J},\bar{K})$.

Case (ii): $n > 1$. If $f \in B^n(G,J,E)$, then $f = dg$, $g \in C^{n-1}(G,J,E)$. Now $g \in C^{n-1}(G,J,E)$ and $dg \in C^n(G,J,E)$ imply by step (2) that $g(G^{n-1}) \subset \bar{K}$. We consider $\beta(g) = \bar{g}$. By step (1), $\bar{g} \in C^{n-1}(\bar{G},\bar{J},\bar{K})$. We check easily that $\beta(f) = \bar{f} = d\bar{g}$. Since by step (1) $\bar{f} \in C^n(\bar{G},\bar{J},\bar{K})$ we get that $\beta(f) \in B^n(\bar{G},\bar{J},\bar{K})$. Let $h \in B^n(\bar{G},\bar{J},\bar{K})$ and $h = dg$, $g \in C^{n-1}(\bar{G},\bar{J},\bar{K})$. Let $g_1 = g \circ \alpha_{n-1}$. $g_1 \in C^{n-1}(G,J,E)$. Put $h_1 = dg_1$. We have by direct calculation $h_1 = h \circ \alpha_n$ and so $h_1 \in C^n(G,J,E)$ and hence to $B^n(G,J,E)$. We then check easily that $\beta(h_1) = h$.

Hence $\beta(B^n(G,J,E)) = B^n(\bar{G},\bar{J},\bar{K})$.

Step (5): β is an isomorphism of $Z^n(G,J,E)$ onto $Z^n(\bar{G},\bar{J},\bar{K})$ and maps $B^n(G,J,E)$ onto $B^n(\bar{G},\bar{J},\bar{K})$. Hence $H^n(G,J,E) \simeq H^n(\bar{G},\bar{J},\bar{K})$. That $H^n(G,J,E^*) \simeq H^n(\bar{G},\bar{J},\bar{K}^*)$ follows similarly.

Theorem 3.2. *Let E, K, G, be such that $\bar{K}$ is Galois closed. Then for all $n \geq 0$ we have*

1) $H^n(G,J,E) \simeq H^n(S,\tau_0,\bar{K})$ and

2) $H^n(G,J,E^*) \simeq H^n(S,\tau_0,\bar{K}^*)$, where $S = \{\sigma \in G(\bar{K}/K) : \sigma$ is extendable to an automorphism of E over $K\}$ and τ_0 is the Krull topology on S.

Proof. By Theorem 2.6 the transformation system $(S,\tau_0,\bar{K})$ $((S,\tau_0,\bar{K}^*))$ is isomorphic to the system $(\bar{G},\bar{J},\bar{K})((\bar{G},\bar{J},\bar{K}^*))$. Hence Proposition 1.8 and Theorem 3.1 establish Theorem 3.2 if $n > 1$. If $n = 0$ we have $H^0(G,J,E) = K = H^0(S,\tau_0,\bar{K})$ and $H^0(G,J,E^*) = K^* = H^0(S,\tau_0,\bar{K})$, since every $\sigma \in G$ gives rise to an element $\sigma \mid \bar{K}$ of S.

This completes the proof of the theorem.

Theorem 3.3. *Let E, K, G be such that $\bar{K}$ is Galois closed and $(\bar{G},\bar{J})$ is a compact space. Then for all $n > 0$*

$$H^n(G,J,E) \simeq H^n(G(\bar{K}/K),\tau_0,\bar{K}) \text{ and}$$

$$H^n(G,J,E^*) \simeq H^n(G(\bar{K}/K),\tau_0,\bar{K}^*).$$

Proof. This follows easily from Theorem 3.2 and Theorem 2.7.

Theorem 3.4. *If E, K, G is such that $\bar{K}$ is Galois closed, then $H^1(G, J, E) = 0$ and $H^1(G, J, E^*) = 1$.*

Proof. Consider the case $H^1(G, J, E^*)$. Let $f : G \to E^*$ be a 1-cocycle. Then f is a continuous map from (G, J) into the discrete space E^* such that for all s, $t \in G$, $f(st) = sf(t) \cdot f(s)$.

Hence we get $f(e) = 1$. Then $V = f^{-1}(1)$ must be a open and closed set in (G, J) containing e. Then $G \setminus V$ being also open and closed is a union of cosets of G_0. Hence by Theorem 2.4 V contains a G_α, $\alpha \in \bar{K}$ such that $K(\alpha)$ is finite normal over K. The fixed field of G_α is $K(\alpha)$ since $K(\alpha)$ is finite over K and hence by a theorem of Kaplansky [1, p. 15] s Galois closed. If $s \in G_\alpha$ and $t \in G$ we have $f(st) = sf(t) \cdot f(s) = sf(t)$. But G_α is a normal subgroup of G since $K(\alpha)$ is normal over K.

Hence $st = ts'$, $s' \in G_\alpha$. Then $f(ts') = tf(s') \cdot f(t) = f(t)$. So $f(st) = sf(t) = f(t)$. Thus $f(t) \in K(\alpha)$. Hence $f(G) \subset K(\alpha)$. Also G/G_α can be considered as the Galois group of $K(\alpha)$ over K and if we define $\bar{f}(tG_\alpha) = f(t)$ we have a 1-cycle from G/G_α into $K(\alpha)^*$. Hence by classical Galois cohomology there is an $a \in K(\alpha)^*$ such that

$$\bar{f}(tG_\alpha) = \frac{(tG_\alpha)(a)}{a} = \frac{t(a)}{a}.$$

We now have if $t \in G$, $f(t) = \bar{f}(tG_\alpha) = t(a)/a$ and hence f is a coboundary. But by hypothesis $f : G \to E^*$ is continuous. Hence it follows that $f \in B^1(G, J, E^*)$. Hence $H^1(G, J, E^*) = 1$. That $H^1(G, J, E) = 0$ follows similarly.

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